IL NUOVO CIMENTO VOL. LX B, N. 2 11 Aprile 1969

Junction Conditions for Spherically Symmetric Matter in Co-moving Co-ordinates.

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(ricevuto il 5 Agosto 1968)

Summary. — The conditions ensuring the possibility of going over to admissible co-ordinates (in the Lichnerowicz sense) are established for the case of co-moving co-ordinates. It is proved that the Oppenheimer-Snyder solution is correctly matched and does not need modifications. Attempted modifications are shown to be incorrect.

1. – Introduction.

Much has been done by LICHNEROWICZ (1), in the general case, and by ISRAEL (2), in the spherically symmetric case, to elucidate the problem of junction conditions which had been studied previously by SYNGE and O'BRIEN (3).

From the mathematical point of view, the most simple and satisfactory expression for the matching conditions is, following Linchnerowicz, the assumption that there exists a system of co-ordinates called admissible in which the metric tensor satisfies the continuity conditions

$$g_{ik} = (C^1_\Sigma, C^3) \tag{1}$$

(the piecewise continuity of the metric up to the third derivative in a finite number of subdomains and the continuity of the metric and its first derivatives across each 3-space separating two subdomains).

(1) A. LICHNEROWICZ: *Théories relativistes de la gravitation et de l'électromagnetisme* (Paris, 1955).

(2) W. ISRAEL: *Proc. Roy. Soc.*, A **248**, 404 (1958).

(3) S. O'BRIEN and J. L. SYNGE: *Jump conditions at discontinuities in general relativity*, in *Comm. Dublin Inst. Adv. Stud.*, A **9** (1952).

However, whenever these continuity conditions are not satisfied at a given boundary, we face *a priori* two possibilities: either the matching at this boundary is incorrect or the matching is correct but the co-ordinates are not admissible.

It is possible in principle to find out which one of the two possbilities is the correct one by going over to Gaussian co-ordinates; Lichnerowicz has shown that Gaussian co-ordinates are admissible so that expressed in these co-ordinates it is always possible to check the correctness of the junction conditions (1).

However, it is not always easy to go over to Gaussian co-ordinates; solutions are often expressed in a simpler way in non-Gaussian co-ordinates; to go over to Gaussian co-ordinates we have to solve a system of partial-differential equations which, more often than not, are very complicated.

ISRAEL (2) gives the junction conditions for the g_{ik} in the case of curvature co-ordinates, *i.e.* for which the line element is of the form

$$ds^2 = -A(r,t)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + B(r,t)\,dt^2 \tag{2}$$

(these co-ordinates are derivable from admissible ones by a C^1-transformation and are therefore not admissible); as a result, ISRAEL establishes the junction conditions

$$(\Delta\dot{g}_{11})^2 = \Delta g'_{11}\Delta g'_{44}\,, \tag{3}$$

$$(\Delta g'_{44})^2 = \Delta\dot{g}_{11}\Delta\dot{g}'_{44}\,, \tag{4}$$

which are weaker than (1); they are for instance satisfied if we take $\Delta\dot{g}_{11} = 0$, $\Delta g'_{44} = 0$, $\Delta g'_{11} \neq 0$, $\Delta\dot{g}_{44} \neq 0$.

However, when working for instance with co-moving co-ordinates ($T^1_4 = 0$) it is not known *a priori* if the co-ordinates are derivable from admissible ones by a C^1- or a C^0-transformation. This has given rise to controversy as to the correctness of the Oppenheimer-Snyder solution (in which $\Delta g_{11} \neq 0$) and to attempts by HOYLE-NARLIKAR (4), MCVITTE (5) and NARIAI (6) to propose different solutions.

This paper deals with the conditions to be imposed on the metric in the co-moving case so as to ensure the existence of an admissible system of co-ordinates obtainable by a co-ordinate transformation.

2. – Assumptions.

We shall not assume the continuity of the metric g_{ik} at the junction hypersurface; we shall however assume:

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(4) F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **278**, 465 (1964).
(5) G. C. MCVITTIE: *Astrophys. Journ.*, **140**, 401 (1964).
(6) H. NARIAI: *Progr. Theor. Phys.*, **34**, 173 (1965).

1) The continuity of the co-ordinates x^i across the junction hypersurface.

2) That the induced 3-metric on the junction hypersurface is the same for both sides of the hypersurface (it can be shown that this assumption is implied in assumption 1)).

3) The junction hypersurface has for equation $r = R = \text{const}$, the same for both sides of the hypersurface.

Assumption 2) means that the junction 2-surface is at rest in our system of co-ordinates.

4) We shall suppose that the metric is of the form

$$-A^+ \, \mathrm{d}r^2 - B^+(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) + C^+ \, \mathrm{d}t^2 , \tag{5}$$

$$-A^- \, \mathrm{d}r^2 - B^-(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) + C^- \, \mathrm{d}t^2 , \tag{6}$$

the signs $+$ and $-$ indicating the solution on one side or another of the junction hypersurface.

Any co-moving spherically symmetric solution can be put into this form. In fact a change of co-ordinates can be made with two arbitrary functions $r = r(\bar{r}, \bar{t})$; $t = t(\bar{r}, \bar{t})$ which allows us therefore to impose two conditions for determining these functions. We can therefore always impose as conditions the co-moving one $T^1_4 = 0$ as well as the vanishing of the coefficient of $\mathrm{d}r\,\mathrm{d}t$ in the metric tensor.

3. – The junction conditions.

The induced metric on the hypersurface $r = R$ by the two solutions (5) and (6) is

$$-B^+ \, \mathrm{d}\Omega^2 + C^+ \mathrm{d}t^2 , \tag{7}$$

$$-B^- \, \mathrm{d}\Omega^2 + C^- \mathrm{d}t^2 \tag{8}$$

(with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2$), therefore we must have on the hypersurface, according to assumption 1):

$$[B^+ = B^-] , \qquad [C^+ = C^-] , \tag{9}$$

(an equation between square brackets will indicate that the quantities are calculated at the hypersurface $r = R$).

Applying the transformation

$$r = p(\bar{r}, \bar{t}) , \qquad t = q(\bar{r}, \bar{t}) \tag{10}$$

on the $-$ metric (6) and the transformation

$$r = f(\bar{r}, \bar{t}) \,, \qquad t = g(\bar{r}, \bar{t}) \tag{11}$$

on the $+$ metric (5), we go over to normal Gaussian co-ordinates which are characterized in our case by

$$g_{11} = 1 \,, \qquad g_{14} = 0 \,. \tag{12}$$

In the new system of co-ordinates we can impose that the junction hypersurface be also characterized by $r = R$. The condition $g_{11} = 1$ can be written

$$A^+ f_1^2 - C^+ g_1^2 = 1 \,, \tag{13}$$

$$A^- p_1^2 - C^- q_1^2 = 1 \tag{14}$$

(the lower indices indicate partial differentiation).

The condition $g_{14} = 0$ leads to

$$A^+ f_1 f_2 - C^+ g_1 g_2 = 0 \,, \tag{15}$$

$$A^- p_1 p_2 - C^- q_1 q_2 = 0 \tag{16}$$

We have therefore four partial-differential equations to determine the four unknown functions f, g, p and q. The Cauchy initial data on the junction hypersurface may be determined from (10) and (11); we obtain

$$f(R, \bar{t}) = R \,, \tag{17}$$

$$g(R, \bar{t}) = \bar{t} \,, \tag{18}$$

$$p(R, \bar{t}) = R \,, \tag{19}$$

$$q(R, \bar{t}) = \bar{t} \tag{20}$$

(eqs. (18) and (20) may always be imposed by a transformation of the time co-ordinate not involving the radial co-ordinate).

The four partial-differential equations with their Cauchy data should determine the solution; however, the Lichnerowicz conditions impose the continuity of the metric coefficients and their first derivatives in the case of Gaussian co-ordinates. We must therefore impose the following:

$$\left[\frac{\partial \bar{B}^+}{\partial \bar{r}} = \frac{\partial \bar{B}^-}{\partial \bar{r}}\right], \qquad [\bar{C}^+ = \bar{C}^-], \qquad \left[\frac{\partial \bar{C}^+}{\partial \bar{r}} = \frac{\partial \bar{C}^-}{\partial \bar{r}}\right].$$

These conditions may be written as follows:

(21) $$[B^+ f_1 + B^+ g_1 = B^- p_1 + B^- q_1]\,,$$

(22) $$[C^+ g_2^2 - A^+ f_2^2 = C^- q_2^2 - A^- p_2^2]\,,$$

(23) $$[C_1^+ g_2^2 + 2C^+ g_2 g_{21} - A_1^+ f_2^2 - 2A^+ f_2 f_{21} = C_1^- q_2^2 + 2C^- q_2 q_{21} - A_1^- p_2^2 - 2A^- p_2 p_{21}]\,.$$

With the additional conditions (21), (22) and (23), the problem of determining the four functions f, g, p and q becomes over-determined so that we must find out the compatibility conditions. Let us remark that the Cauchy data (17) and (19) impose already

(24) $$[f_2 = p_2 = 0]\,,$$

in consequence of which the partial-differential equations (15) and (16) lead to

(25) $$[g_1 = q_1 = 0]$$

(the alternative $[g_2$ or $q_2 = 0]$ is forbidden by (18) and (20)). Moreover, we have from the Cauchy data (18) and (20)

(26) $$[g_2 = q_2 = 1]\,.$$

The additional conditions (21), (22) and (23) become

(27) $$[B_1^+ f_1 = B_1^- p_1]\,,$$

(28) $$[C^+ g_2^2 = C^- q_2^2]\,,$$

(29) $$[C_1^+ g_2^2 = C_1^- q_2^2]\,, \qquad \text{or } [C_1^+ = C_1^-]\,.$$

Equation (28) is always satisfied and does not represent an additional condition.

Taking into account that from (13) and (14) we have

(30) $$[A^+ f_1^2 = 1]\,, \qquad \text{and} \qquad [A^- p_1^2 = 1]\,,$$

we may write instead of (27)

(31) $$[(B_1^+)^2/A^+ = (B_1^-)^2/A^-]\,.$$

Conditions (29) and (31) are necessary for the compatibility of the partial-differential equation (13), (14), (15) and (16) and their Cauchy data (17), (18), (19) and (20) with the additional conditions (21), (22) and (23); the sufficiency of (29) and (31) is obvious if we see that once they are satisfied, the additional conditions (21), (22) and (23) become mere consequences of the partial-differential equations and their Cauchy data. We may therefore state:

Two metrics of the form (5) and (6) joined at an hypersurface of equation $r = R = \text{const}$ and having the same induced metric on the hypersurface can be transformed to an admissible system of co-ordinates if the conditions (29) and (31) are satisfied; these conditions are necessary and sufficient.

Since the identity of the induced metric implies the continuity of B and C, the junction conditions are in our case

$$[B^+ = B^-]\,,\qquad [C^+ = C^-]\,,\qquad [C_1^+ = C_1^-]\,,\qquad [(B_1^+)^2/A^+ = (B_1^-)^2/A^-]\,. \tag{32}$$

4. – The matching problem in the Oppenheimer-Snyder solution.

These results allow us to settle the controversy that has been raised [7] about the Oppenheimer-Snyder [8] solution given by

$$B^+ = \left[-\frac{3}{2}(2m)^{\frac{1}{2}}\left(\frac{r}{R}\right)^{\frac{3}{2}} t + r^{\frac{3}{2}}\right]^{\frac{4}{3}}, \tag{33}$$

$$A^+ = (B_1^+)^2/4B^+\,, \tag{34}$$

$$C^+ = 1\,, \tag{35}$$

$$B^- = \left[-\frac{3}{2}(2m)^{\frac{1}{2}} t + r^{\frac{3}{2}}\right]^{\frac{4}{3}}, \tag{36}$$

$$A^- = (B_1^-)^2/4B^-\,, \tag{37}$$

$$C^- = 1\,. \tag{38}$$

It is seen by inspection that all the junction conditions (32) are satisfied; we can therefore say that the Oppenheimer-Snyder solution is a correctly matched solution in the Lichnerowicz sense.

The same cannot be said about the Hoyle-Narlikar [4] procedure. They apply on the co-moving solution a transformation of co-ordinates that destroys the co-moving character of the solution in order to be able to match it to the static form of Schwarzschild's exterior solution. In such a case it is not enough to

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([7]) H. NARIAI: *Progr. Theor. Phys.*, **34**, 155 (1965).

([8]) J. R. OPPENHEIMER and H. SNYDER: *Phys. Rev.*, **56**, 455 (1939).

ensure the continuity of the metric but we have to secure two additional conditions that have been established by ISRAEL [2]. These conditions that are weaker than the requirement of continuity for the first derivatives are not fulfilled in the case of the Hoyle-Narlikar solution.

As to McVITTE [5], he goes over to nonorthogonal co-ordinates in which he succeeds to secure the continuity of the metric coefficients and their space derivatives. However, unlike what is the case in the co-moving orthogonal co-ordinates, it is necessary here to secure the continuity of the time derivatives of the metric coefficients. Not having done this, McVitte's solution is not correctly matched.

Remark. The co-moving condition $T^1_4 = 0$ has not been used in the derivation of the junction conditions (32); therefore, these junction conditions are valid in all spherically symmetric cases in which the junction surface is at rest (including of course the co-moving case).

RIASSUNTO (*)

Si stabiliscono le condizioni che assicurano la possibilità di passare alle coordinate ammissibili (nel senso di Lichnerowicz) nel caso di coordinate che si muovano assieme. Si dimostra che la soluzione di Oppenheimer e Snyder è corretta e che non sono necessarie delle modifiche. Si dimostra che le modifiche tentate non sono corrette.

(*) *Traduzione a cura della Redazione.*

Условия сшивки для сферически симметричного вещества в сопутствующих координатах.

Резюме (*). — Для случая сопутствующих координат устанавливаются условия, обеспечивающие возможность перехода к допустимым координатам (в смысле Лишнеровича). Доказывается, что решение Оппенгеймера-Снидера являются корректно подобранными и не нуждаются в изменениях. Показывается, что предпринятые видоизменения являются некорректными.

(*) *Переведено редакцией.*